

ITERATES OF SYSTEMS OF OPERATORS IN SPACES OF ω -ULTRADIFFERENTIABLE FUNCTIONS

C. BOITI*, R. CHAÏLI**, AND T. MAHROUZ**

ABSTRACT. Given two systems $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ of linear partial differential operators with constant coefficients, we consider the spaces \mathcal{E}_ω^P and \mathcal{E}_ω^Q of ω -ultradifferentiable functions with respect to the iterates of the systems P and Q respectively. We find necessary and sufficient conditions, on the systems and on the weights $\omega(t)$ and $\sigma(t)$, for the inclusion $\mathcal{E}_\omega^P \subseteq \mathcal{E}_\sigma^Q$. As a consequence we have a generalization of the classical Theorem of the Iterates.

Keywords: Iterates of systems of operators, Theorem of the Iterates, ultradifferentiable functions.

2010 Mathematics Subject Classification: Primary: 35E20; Secondary: 46E10, 35H99.

1. Introduction

The problem of iterates was first introduced by Komatsu [K1] in the 60's, when he characterized analytic functions u on an open subset $\Omega \subseteq \mathbb{R}^n$ in terms of the behaviour of successive iterates $P^j(D)u$ for a linear partial differential elliptic operator $P(D)$ with constant coefficients. He proved that, if $P(D)$ is an elliptic operator of order m , then a C^∞ function u is real analytic in Ω if and only if for every compact $K \subset\subset \Omega$ there is a constant $C > 0$ such that

$$(1.1) \quad \|P^j(D)u\|_{L^2(K)} \leq C^{j+1}(j!)^m, \quad \forall j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

This is known as the Theorem of the Iterates.

Moreover, the condition that $P(D)$ is elliptic is sufficient and also necessary (cf. [M], [LW]) for the above mentioned result, so that, given a linear partial differential operator $P(D)$ of order m with constant coefficients, the ellipticity growth condition

$$(1.2) \quad |\xi|^{2m} \leq C(1 + |P(\xi)|^2), \quad \forall \xi \in \mathbb{R}^n,$$

for a constant $C > 0$, is equivalent to the equality

$$\mathcal{A}(\Omega) = \mathcal{A}^P(\Omega),$$

where $\mathcal{A}(\Omega)$ is the space of real analytic functions on Ω and $\mathcal{A}^P(\Omega)$ is the space of real analytic functions on Ω with respect to the iterates of P , i.e. the space of C^∞ functions u on Ω satisfying (1.1).

This problem was generalized by Newberger and Zielezny [NZ] to the class of Gevrey functions proving, more in general, that, for a pair of hypoelliptic linear partial differential operators $P(D)$ and $Q(D)$ with constant coefficients, of order m and r respectively, the condition that

$$(1.3) \quad |Q(\xi)|^2 \leq C(1 + |P(\xi)|^2)^h, \quad \forall \xi \in \mathbb{R}^n,$$

*Partially supported by the “INdAM-GNAMPA Project 2015”.

**Supported by “Laboratoire d’analyse mathématique et applications, Université d’Oran 1, Algérie”.

for some $h > 0$, is equivalent to an inclusion of the form

$$\mathcal{E}_{\{t^{\frac{1}{s}}\}}^P(\Omega) \subseteq \mathcal{E}_{\{t^{\frac{r}{smh}}\}}^Q(\Omega)$$

if s is large enough, where $\mathcal{E}_{\{t^{1/s}\}}^P(\Omega)$ is the space of Gevrey functions of order s with respect to the iterates of $P = P(D)$, as defined in (2.4) for a Gevrey weight $\omega(t) = t^{1/s}$.

This result was generalized to the class of ω -ultradifferentiable functions in the sense of [BMT] by [JH], and was considered in the case of systems of operators in the Gevrey setting by [BC1]. Here we implement both papers [JH] and [BC1], considering the case of systems in the spaces of ω -ultradifferentiable functions.

In Section 2 we define the spaces of ω -ultradifferentiable functions $\mathcal{E}_\omega^P(\Omega)$ with respect to the iterates of the system $P = (P_j(D))_{j=1}^N$, both in the Beurling and in the Roumieu setting.

In Section 3 we prove that, given two systems $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ of order m and r respectively, the condition

$$\sum_{j=1}^M |Q_j(\xi)| \leq C \left(1 + \sum_{j=1}^N |P_j(\xi)| \right)^h, \quad \forall \xi \in \mathbb{R}^n,$$

is necessary and sufficient for an inclusion of the form

$$\mathcal{E}_{\omega'}^P(\Omega) \subseteq \mathcal{E}_{\sigma'}^Q(\Omega),$$

under assumptions weaker than hypoellipticity (condition (\mathcal{H}) for the sufficiency in Theorem 3.8 and condition (\mathcal{C}) for the necessity in Theorem 4.4), where $\sigma'(t) = \omega'(t^{\frac{r}{mh}})$ with $\omega'(t) = \omega(t^{1/s})$ and s large enough, both in the Beurling and in the Roumieu setting, for a non-quasianalytic weight ω .

In particular, if $P = (P_j(D))_{j=1}^N$ is an elliptic system, we obtain the Theorem of the Iterates (see Corollary 3.10), i.e.

$$\mathcal{E}_{\omega'}^P(\Omega) = \mathcal{E}_{\omega'}(\Omega).$$

Moreover, we prove that the ellipticity of the system P is also necessary (see Corollary 4.6).

In Example 3.11 we have an application of the above results.

Let us finally recall that the Theorem of the Iterates has also been generalized to the case of variable coefficients, for a single elliptic operator $P(x, D)$. It has been proved in the class of real analytic functions by Kotake and Narasimhan [KN], in the case of Denjoy-Carleman classes of Roumieu type by Lions and Magenes [LM] and of Beurling type with some loss of regularity with respect to the coefficients by Oldrich [O], in the classes of ω -ultradifferentiable functions of Roumieu type, or of Beurling type but with some loss of regularity with respect to the coefficients, by Boiti and Jornet [BJ3].

For a microlocal version of the Theorem of the Iterates see, for instance, [BCM], [BJJ], [BJ1], [BJ2]. For anisotropic Gevrey classes we refer to [Z], [BC2].

2. Spaces of ω -ultradifferentiable functions with respect to the iterates of a system of operators

Let us first recall, from [BMT], the notion of weight functions and of spaces of ω -ultradifferentiable functions of Beurling and Roumieu type:

Definition 2.1. *A non-quasianalytic weight function is a continuous increasing function $\omega : [0, +\infty) \rightarrow [0, +\infty)$ with the following properties:*

$$(\alpha) \exists L > 0 \text{ s.t. } \omega(2t) \leq L(\omega(t) + 1) \quad \forall t \geq 0;$$

- (β) $\int_1^{+\infty} \frac{\omega(t)}{t^2} dt < +\infty$;
- (γ) $\log t = o(\omega(t))$ as $t \rightarrow +\infty$;
- (δ) $\varphi_\omega(t) := \omega(e^t)$ is convex.

For $z \in \mathbb{C}^n$ we write $\omega(z)$ for $\omega(|z|)$, where $|z| = \sum_{j=1}^n |z_j|$. We write φ for φ_ω when it is clear from the context.

Remark 2.2. Condition (β) is the condition of non-quasianalyticity and it will ensure the existence of ω -ultradifferentiable functions with compact support.

In the Beurling setting, condition (γ) may be weakened (cf. [BG]), by the following:

$$(\gamma)' \quad \exists a \in \mathbb{R}, b > 0 : \omega(t) \geq a + b \log(1+t), \quad \forall t \geq 0.$$

The *Young conjugate* φ^* of φ is defined by

$$\varphi^*(s) := \sup_{t \geq 0} \{st - \varphi(t)\}, \quad s \geq 0.$$

Assuming, without any loss of generality, that ω vanishes on $[0, 1]$, we have that φ^* has only non-negative values, it is convex and increasing, $\varphi^*(0) = 0$, $\varphi^*(s)/s$ is increasing and $(\varphi^*)^* = \varphi$ (cf. [BMT]).

An easy computation shows that, for every $a > 0$:

$$(2.1) \quad \sigma(t) = \omega(t^a) \quad \Rightarrow \quad \varphi_\sigma^*(s) = \varphi_\omega^*(s/a).$$

For a compact set $K \subset \mathbb{R}^n$ and $\lambda > 0$ we consider the semi-norm

$$p_{K,\lambda}(u) = \sup_{\alpha \in \mathbb{N}_0^n} \sup_{x \in K} |D^\alpha u(x)| e^{-\lambda \varphi^*(\frac{|\alpha|}{\lambda})}.$$

Then

$$\mathcal{E}_{\omega,\lambda}(K) := \{u \in C^\infty(K) : p_{K,\lambda}(u) < +\infty\}$$

is a Banach space endowed with the norm $p_{K,\lambda}$.

Let us then recall from [BMT] the definition of the space of ω -ultradifferentiable functions of Beurling type in an open set $\Omega \subseteq \mathbb{R}^n$:

$$\mathcal{E}_{(\omega)}(\Omega) := \varprojlim_{K \subset \subset \Omega} \varprojlim_{\lambda > 0} \mathcal{E}_{\omega,\lambda}(K).$$

This is a Fréchet space.

The space of ω -ultradifferentiable functions of Roumieu type is defined by

$$\mathcal{E}_{\{\omega\}}(\Omega) := \varprojlim_{K \subset \subset \Omega} \varinjlim_{m \in \mathbb{N}} \mathcal{E}_{\omega, \frac{1}{m}}(K).$$

Let us now consider a system $P = (P_j(D))_{j=1}^N$ of linear partial differential operators with constant coefficients. For $\beta \in \mathbb{N}_0^N$ we define the iterates of the system P as

$$P^\beta := P_1^{\beta_1}(D) \circ P_2^{\beta_2}(D) \circ \cdots \circ P_N^{\beta_N}(D),$$

where $P_j^{\beta_j}(D)$ is the β_j -th iterate of the operator $P_j(D)$, i.e.

$$P_j^{\beta_j}(D) = \underbrace{P_j(D) \circ \cdots \circ P_j(D)}_{\beta_j},$$

and $P^0(D)u = u$.

We shall say, in the following, that the system $P = (P_j(D))_{j=1}^N$ has order m if each operator $P_j(D)$ has order m . In this case, for a compact $K \subset \mathbb{R}^n$ and $\lambda > 0$ we consider the semi-norm

$$p_{K,\lambda}^P(u) := \sup_{\beta \in \mathbb{N}_0^N} \|P^\beta u\|_{L^2(K)} e^{-\lambda \varphi^*\left(\frac{|\beta|m}{\lambda}\right)}$$

and define

$$(2.2) \quad \mathcal{E}_{\omega,\lambda}^P(K) := \{u \in C^\infty(K) : p_{K,\lambda}^P(u) < +\infty\}.$$

For an open set $\Omega \subseteq \mathbb{R}^n$ we define the space of ω -ultradifferentiable functions of Beurling type with respect to the iterates of the system $P = (P_j(D))_{j=1}^N$ by:

$$(2.3) \quad \mathcal{E}_{(\omega)}^P(\Omega) := \varprojlim_{K \subset \subset \Omega} \varprojlim_{\lambda > 0} \mathcal{E}_{\omega,\lambda}^P(K).$$

Analogously, we define the space of ω -ultradifferentiable functions of Roumieu type with respect to the iterates of the system P by:

$$(2.4) \quad \mathcal{E}_{\{\omega\}}^P(\Omega) := \varprojlim_{K \subset \subset \Omega} \varinjlim_{\ell \in \mathbb{N}} \mathcal{E}_{\omega, \frac{1}{\ell}}^P(K).$$

Notation. In the following we shall write $\mathcal{E}_\omega^P(\Omega)$ if the statement holds both in the Beurling case $\mathcal{E}_{(\omega)}^P(\Omega)$ and in the Roumieu case $\mathcal{E}_{\{\omega\}}^P(\Omega)$.

Remark 2.3. When the system is given by a single operator $P = P(D)$, the above defined spaces $\mathcal{E}_\omega^P(\Omega)$ coincide with the corresponding ones defined in [BJJ] (see [JH] for the original, slightly different, definition).

Analogously as in [BC1], we give the following:

Definition 2.4. We say that the system $P = (P_j(D))_{j=1}^N$ satisfies condition (\mathcal{C}) if for every $\lambda > 0$ and $K \subset \subset \Omega$ the space $\mathcal{E}_{\omega,\lambda}^P(K)$ defined in (2.2) is a Banach space endowed with the norm $p_{K,\lambda}^P$.

Let $\{K_\ell\}_{\ell \in \mathbb{N}}$ be a compact exhaustion of Ω , i.e. a sequence of compact subsets of Ω with $K \subset \overset{\circ}{K}_{\ell+1}$ and $\bigcup_\ell K_\ell = \Omega$. We have that

$$(2.5) \quad \mathcal{E}_{(\omega)}^P(\Omega) = \varprojlim_{\ell \in \mathbb{N}} \varprojlim_{m \in \mathbb{N}} \mathcal{E}_{\omega,m}^P(K_\ell) = \varprojlim_{\ell \in \mathbb{N}} \mathcal{E}_{\omega,\ell}^P(K_\ell).$$

Remark 2.5. If condition (\mathcal{C}) is satisfied, then $\mathcal{E}_{(\omega)}^P(\Omega)$, endowed with the metrizable local convex topology defined by the fundamental system of semi-norms $\{p_{K_\ell,\ell}^P\}_{\ell \in \mathbb{N}}$, is a Fréchet space. On the contrary, condition (\mathcal{C}) does not guarantee that $\mathcal{E}_{\{\omega\}}^P(\Omega)$ is complete.

However, if $P = (P_j(D))_{j=1}^N$ is a system of hypoelliptic operators, then it can be proved, as in [JH, Thm. 3.3], that both $\mathcal{E}_{(\omega)}^P(\Omega)$ and $\mathcal{E}_{\{\omega\}}^P(\Omega)$ are complete.

In the case of a single operator $P = P(D)$ it was proved in [JH, Prop. 3.1] that also the converse is valid: if $\mathcal{E}_\omega^P(\Omega)$ is complete, then $P(D)$ must be hypoelliptic. This is not true in the case of systems. Take, for instance, $P = (D_j)_{j=1}^n$ for $D_j = -i\partial_{x_j}$. Then $\mathcal{E}_\omega^P(\Omega) = \mathcal{E}_\omega(\Omega)$ is complete by [BMT, Prop. 4.9], but the operators $P_j(D) = D_j$ are not hypoelliptic.

Remark 2.6. It is possible to construct a finer locally convex topology that makes $\mathcal{E}_\omega^P(\Omega)$ always complete, without any assumption on the operators.

In the Beurling case we take a compact exhaustion $\{K_\ell\}_{\ell \in \mathbb{N}}$ of Ω , set

$$p_\ell(u) := \sup_{|\alpha| \leq \ell} \sup_{x \in K_\ell} |D^\alpha u(x)|$$

and then consider the semi-norm

$$\tau_\ell^P(u) := \max \{p_{K_\ell, \ell}^P(u), p_\ell(u)\}.$$

We have that $\mathcal{E}_{(\omega)}^P(\Omega)$, endowed with the convex topology defined by the fundamental system of semi-norms $\{\tau_\ell^P\}_{\ell \in \mathbb{N}}$, is a Fréchet space. The proof is standard.

In the Roumieu case we consider, for $\ell \in \mathbb{N}$ and $K \subset\subset \Omega$, the fundamental system of semi-norms $\{\tau_{K, \ell, m}^P\}_{m \in \mathbb{N}}$ defined by

$$(2.6) \quad \tau_{K, \ell, m}^P(u) := \max\{p_{K, \frac{1}{\ell}}^P(u), p_m(u)\}.$$

This makes $\mathcal{E}_{\omega, \frac{1}{\ell}}^P(K)$ a Fréchet space. Considering then on $\mathcal{E}_{\{\omega\}}^P(\Omega)$ the topology induced by (2.4), we can prove, as in [JH, Prop. 3.5], that $\mathcal{E}_{\{\omega\}}^P(\Omega)$ is complete.

We now want to look for sufficient and necessary conditions in order to obtain the Theorem of the Iterates for systems $P = (P_j(D))_{j=1}^N$ of linear partial differential operators with constant coefficients in the classes of ω -ultradifferentiable functions.

3. A sufficient condition

Analogously as in [BC1], we give the following:

Definition 3.1. Let $P = (P_j(D))_{j=1}^N$ be a system of linear partial differential operators with constant coefficients of order m . We say that P satisfies condition (\mathcal{H}) if there exist $C > 0$ and $\gamma \geq m$ such that

$$(3.1) \quad \sum_{j=1}^N |P_j^{(\alpha)}(\xi)| \leq C \left(1 + \sum_{j=1}^N |P_j(\xi)|\right)^{1 - \frac{|\alpha|}{\gamma}}, \quad \forall \alpha \in \mathbb{N}_0^n, \xi \in \mathbb{R}^n,$$

where $P_j^{(\alpha)}(\xi) = \partial_\xi^\alpha P_j(\xi)$.

Remark 3.2. If the system $P = (P_j(D))_{j=1}^N$ satisfies condition (\mathcal{H}) for some $\gamma \geq m$, there exists a smallest $\gamma_P \geq m$ such that P satisfies (3.1) for $\gamma = \gamma_P$; moreover $\gamma_P \in \mathbb{Q}$. Indeed, the inequality (3.1) implies that there exists $C' > 0$ such that

$$(3.2) \quad |\text{grad } P_i(\xi)| \leq C' \left(1 + \sum_{j=1}^N |P_j(\xi)|\right)^{1 - \frac{1}{\gamma}}, \quad \forall i = 1, \dots, N.$$

Applying then the Tarski-Seidenberg theorem to the semi-algebraic function

$$M_i(\lambda) = \sup_{\sum_{j=1}^N |P_j(\xi)| = \lambda} |\text{grad } P_i(\xi)|,$$

we can argue as in [H1, Thm. 3.1] to prove that for every $i \in \{1, \dots, N\}$ there exists a smallest γ_i such that

$$(3.3) \quad |P_i^{(\alpha)}(\xi)| \leq C \left(1 + \sum_{j=1}^N |P_j(\xi)|\right)^{1 - \frac{|\alpha|}{\gamma_i}}, \quad \forall \alpha \in \mathbb{N}_0^n, \xi \in \mathbb{R}^n.$$

Then $\gamma_P := \max\{\gamma_1, \dots, \gamma_N\}$ is the smallest γ satisfying (3.1) and moreover $\gamma_P \in \mathbb{Q}$ and $\gamma_P \geq m$.

In the following, for a system P satisfying condition (\mathcal{H}) , we shall always refer to γ_P as defined in Remark 3.2.

Remark 3.3. If $P = P(D)$ is a hypoelliptic operator, then condition (\mathcal{H}) is satisfied because of [H1, Thm. 3.1]. However, in general condition (\mathcal{H}) is weaker than hypoellipticity. Take for instance in \mathbb{R}^2 the operator $P(D) = P(D_1, D_2) = D_1^2$. It is trivially not hypoelliptic, but it satisfies condition (\mathcal{H}) for $\gamma = 2$.

More in general, if $P = (P_j(D))_{j=1}^N$ is a system of hypoelliptic operators, then P satisfies condition (\mathcal{H}) . If the system P is elliptic, i.e.

$$(3.4) \quad |\xi|^m \leq C \left(1 + \sum_{j=1}^N |P_j(\xi)| \right), \quad \forall \xi \in \mathbb{R}^n,$$

then condition (\mathcal{H}) is satisfied for $\gamma_P = m$.

In order to compare, for two given systems $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$, the corresponding spaces $\mathcal{E}_\omega^P(\Omega)$ and $\mathcal{E}_\omega^Q(\Omega)$, we introduce the following:

Definition 3.4. Let $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ be two systems of linear partial differential operators with constant coefficients. If there exist $C, h > 0$ such that

$$(3.5) \quad \sum_{j=1}^M |Q_j(\xi)| \leq C \left(1 + \sum_{j=1}^N |P_j(\xi)| \right)^h, \quad \forall \xi \in \mathbb{R}^n,$$

we say that Q is h -weaker than P and we write $Q \prec_h P$.

Remark 3.5. If $P = P(D)$ and $Q = Q(D)$ are single operators and $P(D)$ is hypoelliptic, then by [H1, Thm. 3.2] there is a smallest h such that Q is h -weaker than P , and moreover $h \in \mathbb{Q}$.

More in general, if $P = (P_j(D))_{j=1}^N$ is h -weaker than $Q = (Q_j(D))_{j=1}^M$, there exists a smallest $h > 0$ such that (3.5) is satisfied and moreover $h \in \mathbb{Q}$. Indeed, we can argue as in [H1, Thm. 3.2] and Remark 3.2, taking the semi-algebraic functions

$$M_i(\lambda) = \sup_{\sum_{j=1}^N |P_j(\xi)| = \lambda} |Q_i(\xi)|.$$

Definition 3.6. If $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ are two systems with $P \prec_h Q$ and $Q \prec_h P$, we say that the systems P and Q are h -equally strong, and we write $P \approx_h Q$.

Remark 3.7. Arguing as in [H1, pg 210], we can easily prove that if $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ are two systems of order m and r respectively, satisfying condition (\mathcal{H}) and 1-equally strong, then $m = r$ and $\gamma_P = \gamma_Q$.

We are now ready to prove the following result:

Theorem 3.8. Let $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ be two systems of linear partial differential operators with constant coefficients, of order m and r respectively. Assume that P and Q satisfy condition (\mathcal{H}) of Definition 3.1 and that Q is h -weaker than P . Let Ω be an open subset of \mathbb{R}^n . Let ω be a non-quasianalytic weight function and set $\omega'(t) = \omega(t^{1/s})$ for $s \geq \gamma_P/m$. Then

$$(3.6) \quad \mathcal{E}_{(\omega')}^P(\Omega) \subseteq \mathcal{E}_{(\sigma')}^Q(\Omega)$$

$$(3.7) \quad \mathcal{E}_{\{\omega'\}}^P(\Omega) \subseteq \mathcal{E}_{\{\sigma'\}}^Q(\Omega),$$

for $\sigma'(t) = \omega'(t^{\frac{r}{mh}}) = \omega(t^{\frac{r}{smh}})$.

Proof. Beurling case:

Let $u \in \mathcal{E}_{(\omega')}^P(\Omega)$. For every compact $K \subset \Omega$ there exist an open set F relatively compact in Ω and $\delta > 0$ such that

$$K \subset F_{(M+1)\delta} \subset F \subset \Omega,$$

where

$$F_\sigma := \{x \in F : d(x, \partial F) > \sigma\}.$$

Moreover, for every $q \in \mathbb{N}$ there exists $C_q > 0$ such that

$$(3.8) \quad \sum_{|\beta|=\ell} \|P^\beta u\|_{L^2(F)} \leq C_q e^{q\varphi_\omega^*\left(\frac{\ell m}{q}\right)} = C_q e^{q\varphi_\omega^*\left(\frac{\ell m s}{q}\right)}, \quad \forall \ell \in \mathbb{N},$$

by the definition of $\mathcal{E}_{(\omega')}^P(\Omega)$ and by (2.1).

By assumption $Q \prec_h P$ and, by Remark 3.5, there exists $\mu, \nu \in \mathbb{N}$ such that $h = \mu/\nu$.

Arguing as in [BC1, Thm. 2.4], we fix $\alpha \in \mathbb{N}_0^M$, choose $k_j, \ell_j \in \mathbb{N}_0$ such that $\alpha_j = k_j \nu + \ell_j$, with $\ell_j \leq \nu - 1$, for $1 \leq j \leq M$, and set $k = \sum_{j=1}^M k_j$. From [BC1, formula (2.12)] there exist $C_1, C_2 > 0$ such that for every $u \in C^\infty(F)$:

$$(3.9) \quad \|Q^\alpha u\|_{L^2(F_{(M+1)\delta})} \leq C_1^M \left[\sum_{i=0}^M \binom{M}{i} M^i C_2^{k+i} \sum_{|\beta| \leq k+i} \binom{k+i}{|\beta|} \cdot \left(\frac{k+i}{\delta}\right)^{(k+i-|\beta|)\gamma_P \mu} \|P^{\beta\mu} u\|_{L^2(F)} \right].$$

If $\gamma_P \leq sm$, then from (3.8) we have, for all $\ell \leq k$:

$$(3.10) \quad \begin{aligned} k^{(k-\ell)\gamma_P \mu} \sum_{|\beta|=\ell} \|P^{\beta\mu} u\|_{L^2(F)} &\leq C_q k^{(k-\ell)sm\mu} e^{q\varphi_\omega^*\left(\frac{m\ell\mu s}{q}\right)} \\ &\leq C_q \left(1 + \frac{\ell}{k-\ell}\right)^{\frac{k-\ell}{\ell}sm\mu\ell} (k-\ell)^{(k-\ell)sm\mu} e^{q\varphi_\omega^*\left(\frac{m\ell\mu s}{q}\right)} \\ &\leq C_q e^{sm\mu\ell} [(k-\ell)sm\mu]^{(k-\ell)sm\mu} e^{q\varphi_\omega^*\left(\frac{m\ell\mu s}{q}\right)}. \end{aligned}$$

Since $\omega(t)$ is a non-quasianalytic weight function, condition (β) implies $\omega(t) = o(t)$ and hence for every $q' \in \mathbb{N}$ there exists $C_{q'} > 0$ such that from [AJO, Rem. 2.4]:

$$(3.11) \quad y \log y \leq y + q'\varphi_\omega^*\left(\frac{y}{q'}\right) + C_{q'}, \quad \forall y > 0.$$

Applying the above inequality to (3.10) we have that

$$(3.12) \quad k^{(k-\ell)\gamma_P \mu} \sum_{|\beta|=\ell} \|P^{\beta\mu} u\|_{L^2(F)} \leq C_q e^{sm\mu\ell} e^{(k-\ell)sm\mu} e^{q'\varphi_\omega^*\left(\frac{(k-\ell)m\mu s}{q'}\right)} e^{C_{q'}} e^{q\varphi_\omega^*\left(\frac{m\ell\mu s}{q}\right)}.$$

By condition (α) of Definition 2.1 there exists $\tilde{L} > 0$ such that

$$\omega(et) \leq \tilde{L}(1 + \omega(t)), \quad \forall t \geq 0.$$

Then, from [BJ3, Prop. 21(e) and Rem. 22] we have that for every $\rho, \lambda > 0$ there exists $\lambda', D_{\rho, \lambda} > 0$ such that

$$(3.13) \quad \rho^j e^{\lambda\varphi_\omega^*(j/\lambda)} \leq D_{\rho, \lambda} e^{\lambda'\varphi_\omega^*(j/\lambda')}, \quad \forall j \in \mathbb{N}_0,$$

with $\lambda' = \lambda/\tilde{L}^{\lfloor \log \rho + 1 \rfloor}$ and $D_{\rho, \lambda} = \exp\{\lambda \lfloor \log \rho + 1 \rfloor\}$, where $\lfloor \log \rho + 1 \rfloor$ is the integer part of $(\log \rho + 1)$.

Applying (3.13) in (3.12) we have that for every $\lambda > 0$ there exists $C_\lambda > 0$ such that

$$(3.14) \quad k^{(k-\ell)\gamma_P\mu} \sum_{|\beta|=\ell} \|P^{\beta\mu}u\|_{L^2(F)} \leq C_\lambda e^{\lambda\varphi_\omega^*\left(\frac{m\ell\mu s}{\lambda}\right)} e^{\lambda\varphi_\omega^*\left(\frac{(k-\ell)m\mu s}{\lambda}\right)}.$$

From condition (α) of Definition 2.1, by [BMT, Lemma 1.2] we have that there exists $L' > 0$ such that

$$\omega(u+v) \leq L'(\omega(u) + \omega(v) + 1), \quad \forall u, v \geq 0,$$

and hence for all $j, k \in \mathbb{N}_0, \lambda > 0$:

$$(3.15) \quad \begin{aligned} e^{\lambda\varphi^*\left(\frac{j}{\lambda}\right) + \lambda\varphi^*\left(\frac{k}{\lambda}\right)} &= \sup_{s \geq 0} e^{js - \lambda\varphi_\omega(s)} \cdot \sup_{t \geq 0} e^{kt - \lambda\varphi_\omega(t)} = \sup_{u, v \geq 1} e^{j \log u + k \log v - \lambda(\omega(u) + \omega(v))} \\ &\leq \sup_{u, v \geq 1} u^j v^k e^{-\frac{\lambda}{L'}\omega(u+v)} e^\lambda \leq e^\lambda \sup_{u, v \geq 1} (u+v)^{j+k} e^{-\frac{\lambda}{L'}\omega(u+v)} \\ &\leq e^\lambda \sup_{\sigma \geq 0} e^{(j+k)\sigma - \frac{\lambda}{L'}\varphi_\omega(\sigma)} = e^\lambda e^{\frac{\lambda}{L'}\varphi_\omega^*\left(\frac{j+k}{\lambda/L'}\right)}. \end{aligned}$$

Applying it to (3.14) we have that for every $\tilde{q} \in \mathbb{N}$ there exists $C_{\tilde{q}} > 0$ such that for all $\ell \leq k$:

$$(3.16) \quad k^{(k-\ell)\gamma_P\mu} \sum_{|\beta|=\ell} \|P^{\beta\mu}u\|_{L^2(F)} \leq C_{\tilde{q}} e^{\tilde{q}\varphi_\omega^*\left(\frac{k\mu ms}{\tilde{q}}\right)}.$$

Substituting in (3.9) we obtain, for some constant $A > 0$:

$$(3.17) \quad \begin{aligned} \|Q^\alpha u\|_{L^2(F_{(M+1)\delta})} &\leq A^k C_{\tilde{q}} e^{\tilde{q}\varphi_\omega^*\left(\frac{(k+M)\mu ms}{\tilde{q}}\right)} \leq A^k C_{\tilde{q}} e^{\frac{\tilde{q}}{2}\varphi_\omega^*\left(\frac{2k\mu ms}{\tilde{q}}\right)} e^{\frac{\tilde{q}}{2}\varphi_\omega^*\left(\frac{2M\mu ms}{\tilde{q}}\right)} \\ &\leq C'_{\tilde{q}} A^{\mu ms k} e^{\frac{\tilde{q}}{2}\varphi_\omega^*\left(\frac{k\mu ms}{\tilde{q}/2}\right)} \leq D_q e^{q\varphi_\omega^*\left(\frac{k\mu ms}{q}\right)} \end{aligned}$$

by the convexity of φ_ω^* and by (3.13), for $q = \tilde{q}/(2\tilde{L}^{\lfloor \log A + 1 \rfloor})$.

Since $k \leq |\alpha|/\nu$ by construction, from (3.17) we thus have that for every $q \in \mathbb{N}$ there exists $D_q > 0$ such that

$$\|Q^\alpha u\|_{L^2(K)} \leq \|Q^\alpha u\|_{L^2(F_{(M+1)\delta})} \leq D_q e^{q\varphi_\omega^*\left(\frac{|\alpha|\mu ms}{\nu q}\right)} = D_q e^{q\varphi_{\sigma'}^*\left(\frac{|\alpha|r}{q}\right)}, \quad \forall \alpha \in \mathbb{N}_0^M,$$

by (2.1), since $\sigma'(t) = \omega(t^{\frac{r}{smh}})$. This proves that $u \in \mathcal{E}_{(\sigma')}^Q(\Omega)$.

Roumieu case:

It is similar to the Beurling case: in (3.8) we take $\frac{1}{q}\varphi_\omega^*(\ell m q)$ instead of $q\varphi_\omega^*(\ell m/q)$ and a fixed constant C instead of C_q , and similarly later on for q', q'', \dots

The proof is complete. \square

Corollary 3.9. *Let $P = (P_j(D))_{j=1}^N$ and $Q = (Q_j(D))_{j=1}^M$ be two systems of order m satisfying condition (\mathcal{H}) and 1-equally strong. Let Ω be an open subset of \mathbb{R}^n . Let ω be a non-quasianalytic weight function and set $\omega'(t) = \omega(t^{1/s})$ for $s \geq \gamma_P/m = \gamma_Q/m$. Then*

$$\mathcal{E}_{(\omega')}^P(\Omega) = \mathcal{E}_{(\omega')}^Q(\Omega) \quad \text{and} \quad \mathcal{E}_{\{\omega'\}}^P(\Omega) = \mathcal{E}_{\{\omega'\}}^Q(\Omega).$$

From Remark 3.3 we obtain the Theorem of the Iterates as a corollary of Theorem 3.8:

Corollary 3.10. *Let $P = (P_j(D))_{j=1}^N$ be an elliptic system of order m . Let Ω be an open subset of \mathbb{R}^n and ω a non-quasianalytic weight function. Then*

$$(3.18) \quad \mathcal{E}_{(\omega)}^P(\Omega) = \mathcal{E}_{(\omega)}(\Omega) \quad \text{and} \quad \mathcal{E}_{\{\omega\}}^P(\Omega) = \mathcal{E}_{\{\omega\}}(\Omega).$$

Proof. Beurling case:

Let us first prove the inclusion

$$(3.19) \quad \mathcal{E}_{(\omega)}^P(\Omega) \subseteq \mathcal{E}_{(\omega)}(\Omega).$$

To this aim we consider the system $Q = (D_j)_{j=1}^n$, for $D_j = -i\partial_{x_j}$. The operators $Q_j(D) = D_j$ are not hypoelliptic, but the system Q satisfies condition (\mathcal{H}) . The system P satisfies (3.4) and hence condition (\mathcal{H}) for $\gamma_P = m$, by Remark 3.3.

Since (3.4) implies that Q is $\frac{1}{m}$ -weaker than P , from Theorem 3.8, with $s = 1 = \gamma_P/m$ and hence $\omega'(t) = \omega(t)$, we have that

$$(3.20) \quad \mathcal{E}_{(\omega)}^P(\Omega) \subseteq \mathcal{E}_{(\sigma)}^Q(\Omega) = \mathcal{E}_{(\sigma)}(\Omega),$$

for $\sigma(t) = \omega(t^{1/(m \cdot \frac{1}{m})}) = \omega(t)$, and hence (3.19) is proved.

Vice versa, since every $P_j(\xi)$ is a polynomial of degree m , we clearly have that P is m -weaker than Q and, from Theorem 3.8,

$$\mathcal{E}_{(\omega)}(\Omega) = \mathcal{E}_{(\omega)}^Q(\Omega) \subseteq \mathcal{E}_{(\sigma)}^P(\Omega)$$

for $\sigma(t) = \omega(t^{\frac{m}{1 \cdot m}}) = \omega(t)$, so that also the opposite inclusion

$$\mathcal{E}_{(\omega)}(\Omega) \subseteq \mathcal{E}_{(\omega)}^P(\Omega)$$

is valid, and hence the equality (3.18) is proved in the Beurling case.

Roumieu case:

The proof is the same as in the Beurling case, using (3.7) instead of (3.6). \square

Example 3.11. Let us consider in \mathbb{R}^2 the system $P = (P_j(D))_{j=1}^2$ defined by

$$P_1(D_1, D_2) = D_1^2, \quad P_2(D_1, D_2) = D_2^2.$$

These operators are not hypoelliptic but the system P satisfies condition (\mathcal{H}) for $\gamma_P = 2$.

Let us then consider $Q = Q(D) = \Delta = -D_1^2 - D_2^2$. This is an elliptic operator of order 2 and hence satisfies condition (\mathcal{H}) for $\gamma_Q = 2$ (see Remark 3.3).

Moreover, P and Q are 1-equally strong and $\gamma_P/m = 1$. We can then apply Corollaries 3.9 and 3.10 with $\omega'(t) = \omega(t)$ and obtain that, for any open subset Ω of \mathbb{R}^2 and for every non-quasianalytic weight function ω :

$$\mathcal{E}_{(\omega)}^P(\Omega) = \mathcal{E}_{(\omega)}^Q(\Omega) = \mathcal{E}_{(\omega)}(\Omega).$$

This means that the elements $u \in \mathcal{E}_{(\omega)}(\Omega)$ can be equivalently determined by estimating their derivatives $D^\alpha u(x) = D_1^{\alpha_1} D_2^{\alpha_2} u(x)$, or the iterates of $Q(D)$, i.e. $\Delta^\beta u(x)$, or the iterates of the system $P = (P_j(D))_{j=1}^2$, i.e. $P^\gamma u(x) = D_1^{2\gamma_1} D_2^{2\gamma_2} u(x)$, for $\alpha, \gamma \in \mathbb{N}_0^2$, $\beta \in \mathbb{N}_0$.

The same holds also in the Roumieu case.

4. A necessary condition

In order to obtain a necessary condition for the inclusions (3.6) or (3.7), we first need to introduce the following:

Definition 4.1. We say that a non-quasianalytic weight function ω satisfies the growth condition B-M-M if there exists a constant $H \geq 1$ such that

$$(4.1) \quad 2\omega(t) \leq \omega(Ht) + H, \quad \forall t \geq 0.$$

Remark 4.2. Condition B-M-M was introduced in [BMM] in order to characterize those weight functions ω for which $\mathcal{E}_{(\omega)}(\Omega)$ (or $\mathcal{E}_{\{\omega\}}(\Omega)$) can be also considered as a Denjoy-Carleman class $\mathcal{E}_{(M_p)}(\Omega)$ (or $\mathcal{E}_{\{M_p\}}(\Omega)$, respectively) as defined in [K2], for some sequence $\{M_p\}$.

Gevrey weights satisfy condition B-M-M.

Let us now prove that the condition $Q \prec_h P$ of Theorem 3.8 is also necessary for the inclusions (3.6) and (3.7).

To this aim we first recall, from [JH, Lemma 4.7], the following:

Lemma 4.3. *For all $h, \lambda > 0$ and $t \geq 1$ we have that:*

$$\begin{aligned} \text{(i)} \quad & \sup_{j \in \mathbb{N}_0} t^j \exp \left\{ -\lambda \varphi^* \left(\frac{hj}{\lambda} \right) \right\} \leq \exp \{ \lambda \omega(t^{1/h}) \}; \\ \text{(ii)} \quad & \sup_{j \in \mathbb{N}_0} t^j \exp \left\{ -\lambda \varphi^* \left(\frac{hj}{\lambda} \right) \right\} \geq \frac{1}{t} \exp \{ \lambda \omega(t^{1/h}) \}. \end{aligned}$$

We can then prove:

Theorem 4.4. *Let Ω be an open subset of \mathbb{R}^n and ω a non-quasianalytic weight function satisfying condition B-M-M. Let $P = (P_j(D))_{j=1}^N$ be a system of linear partial differential operators of order m with constant coefficients satisfying condition (\mathcal{C}) of Definition 2.4 and let $Q = (Q_j(D))_{j=1}^M$ be a generic system of linear partial differential operators of order r with constant coefficients.*

If there exists $h > 0$ such that one of the following inclusions

$$(4.2) \quad \mathcal{E}_{(\omega)}^P(\Omega) \subseteq \mathcal{E}_{(\sigma)}^Q(\Omega)$$

or

$$(4.3) \quad \mathcal{E}_{\{\omega\}}^P(\Omega) \subseteq \mathcal{E}_{\{\sigma\}}^Q(\Omega)$$

holds, for $\sigma(t) = \omega(t^{\frac{r}{mh}})$, then Q is h -weaker than P .

Proof. Roumieu case:

We follow the same ideas of Juan-Huguet in [JH], substituting to the assumption, in [JH, Thm. 4.5], that the single operator $P(D)$ is hypoelliptic, the weaker assumption that the system P satisfies condition (\mathcal{C}) , in the spirit of [BC1].

Let us then assume (4.3) to be satisfied and fix a compact set $K_0 \subset \Omega$.

We have the following inclusions:

$$\mathcal{E}_{(\omega)}^P(\Omega) \subseteq \mathcal{E}_{\{\omega\}}^P(\Omega) \subseteq \mathcal{E}_{\{\sigma\}}^Q(\Omega) = \varprojlim_{K \subset \subset \Omega} \varinjlim_{\ell \in \mathbb{N}} \mathcal{E}_{\sigma, \frac{1}{\ell}}^Q(K) \subseteq \varinjlim_{\ell \in \mathbb{N}} \mathcal{E}_{\sigma, \frac{1}{\ell}}^Q(K_0).$$

By assumption the system P satisfies condition (\mathcal{C}) and hence, by Remark 2.5, $\mathcal{E}_{(\omega)}^P(\Omega)$ is a Fréchet space and $\varinjlim_{\ell \in \mathbb{N}} \mathcal{E}_{\sigma, \frac{1}{\ell}}^Q(K_0)$ is an (LF)-space. We can therefore apply the Closed Graph Theorem and Grothendieck's Factorization Theorem (see [MV, Thms 24.31 and 24.33]) and obtain that there exists $\ell_0 \in \mathbb{N}$ such that

$$\mathcal{E}_{(\omega)}^P(\Omega) \subseteq \mathcal{E}_{\sigma, \frac{1}{\ell_0}}^Q(K_0)$$

with a continuous inclusion.

There exist then a constant $C > 0$, a compact $K \subset \subset \Omega$ and $\lambda > 0$ such that, for all $f \in \mathcal{E}_{(\omega)}^P(\Omega)$:

$$(4.4) \quad \sup_{\beta \in \mathbb{N}_0^M} \|Q^\beta(D)f\|_{L^2(K_0)} e^{-\frac{1}{\ell_0} \varphi_\sigma^*(|\beta|r\ell_0)} \leq C \sup_{\alpha \in \mathbb{N}_0^N} \|P^\alpha(D)f\|_{L^2(K)} e^{-\lambda \varphi_\omega^*(\frac{|\alpha|m}{\lambda})}.$$

For $\xi \in \mathbb{R}^n$, we denote $f_\xi(x) := e^{i\langle x, \xi \rangle}$ and remark that $f_\xi \in \mathcal{E}_{(\omega)}^P(\Omega)$, because for every compact $K \subset\subset \Omega$ and $\lambda > 0$

$$\|P^\alpha(D)f_\xi\|_{L^2(K)} = \|P^\alpha(\xi)f_\xi\|_{L^2(K)} \leq m(K)|P^\alpha(\xi)| \leq C(1 + |\xi|^{m|\alpha|}) \leq C_\xi e^{\lambda' \varphi_\omega^*\left(\frac{|\alpha|m}{\lambda}\right)}$$

for some $C_\xi > 0$ and $\lambda' > 0$, by (3.13). Since $f_\xi \in \mathcal{E}_{(\omega)}^P(\Omega)$ and we can apply (4.4) to f_ξ , obtaining that

$$\sup_{\beta \in \mathbb{N}_0^M} |Q^\beta(\xi)| e^{-\frac{1}{\ell_0} \varphi_\sigma^*(|\beta|r\ell_0)} \leq C' \sup_{\alpha \in \mathbb{N}_0^N} |P^\alpha(\xi)| e^{-\lambda \varphi_\omega^*\left(\frac{|\alpha|m}{\lambda}\right)}$$

for some $C' > 0$.

Therefore

$$\begin{aligned} & \sup_{\beta \in \mathbb{N}_0^M} \left(\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \right)^{|\beta|} e^{-\frac{1}{\ell_0} \varphi_\sigma^*(|\beta|r\ell_0)} \\ & \leq \sup_{\beta \in \mathbb{N}_0^M} \left(\sum_{\beta_1 + \dots + \beta_N = |\beta|} \frac{|\beta|!}{\beta_1! \dots \beta_N!} |Q_1(\xi)|^{\beta_1} \dots |Q_M(\xi)|^{\beta_M} \frac{1}{2^{|\beta|}} e^{-\frac{1}{\ell_0} \varphi_\sigma^*(|\beta|r\ell_0)} \right) \\ & \leq \sup_{\beta \in \mathbb{N}_0^M} |Q^\beta(\xi)| e^{-\frac{1}{\ell_0} \varphi_\sigma^*(|\beta|r\ell_0)} \leq C' \sup_{\alpha \in \mathbb{N}_0^N} |P^\alpha(\xi)| e^{-\lambda \varphi_\omega^*\left(\frac{|\alpha|m}{\lambda}\right)} \\ (4.5) \quad & \leq C'' \sup_{\alpha \in \mathbb{N}_0^N} \left(\sum_{j=1}^N |P_j(\xi)| \right)^{|\alpha|} e^{-\lambda \varphi_\omega^*\left(\frac{|\alpha|m}{\lambda}\right)}. \end{aligned}$$

From Lemma 4.3 it follows that, if $\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \geq 1$ and $\sum_{j=1}^N |P_j(\xi)| \geq 1$, then

$$(4.6) \quad \left(\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \right)^{-1} \exp \left\{ \frac{1}{\ell_0} \sigma \left(\left(\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \right)^{\frac{1}{r}} \right) \right\} \leq \tilde{C} \exp \left\{ \lambda \omega \left(\left(\sum_{j=1}^N |P_j(\xi)| \right)^{\frac{1}{m}} \right) \right\},$$

for some $\tilde{C} > 0$.

From property (γ) of the weight function $\sigma(t)$ we have that (4.6) implies, for some $\lambda' > 0$, if $\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \geq 1$ and $\sum_{j=1}^N |P_j(\xi)| \geq 1$:

$$\exp \left\{ \lambda' \sigma \left(\left(\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \right)^{\frac{1}{r}} \right) \right\} \leq \tilde{C} \exp \left\{ \lambda \omega \left(\left(\sum_{j=1}^N |P_j(\xi)| \right)^{\frac{1}{m}} \right) \right\}.$$

Since $\sigma(t) = \omega\left(t^{\frac{r}{mh}}\right)$ by assumption, we thus obtain:

$$\begin{aligned} & \omega \left(\left(\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \right)^{\frac{1}{mh}} \right) \leq A \left(1 + \omega \left(\left(\sum_{j=1}^N |P_j(\xi)| \right)^{\frac{1}{m}} \right) \right) \\ (4.7) \quad & \leq \omega \left(A' \left(\sum_{j=1}^N |P_j(\xi)| \right)^{\frac{1}{m}} \right) \end{aligned}$$

for some $A' > 0$ if $\sum_{j=1}^N |P_j(\xi)| \geq 1$ and $\sum_{j=1}^M \left| \frac{Q_j}{2}(\xi) \right| \geq 1$, because condition B-M-M implies that for every $k \in \mathbb{N}$ there exists a constant $H_k \geq 1$ such that $2^{k-1}\omega(t) \leq \omega(H_k t)$ for all $t \geq 1$.

Since $\omega(t)$ is increasing, (4.7) implies that there exists a constant $B > 1$ such that

$$(4.8) \quad \sum_{j=1}^M |Q_j(\xi)| \leq B \left(1 + \sum_{j=1}^N |P_j(\xi)| \right)^h,$$

if $\sum_{j=1}^M |\frac{Q_j}{2}(\xi)| \geq 1$ and $\sum_{j=1}^N |P_j(\xi)| \geq 1$.

However, (4.8) is trivial if $\sum_{j=1}^M |\frac{Q_j}{2}(\xi)| \leq 1$ or $\sum_{j=1}^N |P_j(\xi)| \leq 1$, so that (4.8) is satisfied for all $\xi \in \mathbb{R}^n$ and Q is h -weaker than P .

Beurling case:

The proof is similar, but easier, as in the Roumieu case, since $\mathcal{E}_{(\omega)}^P(\Omega)$ and $\mathcal{E}_{(\sigma)}^Q(\Omega)$ are metrizable, and hence the inclusion (4.2) implies (4.4). \square

Remark 4.5. By Remark 2.6, instead of condition (\mathcal{C}) we can consider, in Theorem 4.4, the weaker assumption that $\mathcal{E}_{(\omega)}^P(\Omega)$ is a Fréchet space and then take on $\mathcal{E}_{\sigma,1/\ell}^Q(K_0)$ the fundamental system of semi-norms $\{\tau_{K_0,\ell,m}^Q\}_{m \in \mathbb{N}}$ defined by (2.6), to make $\varinjlim_{\ell \in \mathbb{N}} \mathcal{E}_{\sigma,1/\ell}^Q(K_0)$ an (LF)-space.

As a consequence of Theorem 4.4 we have the converse of Corollary 3.10:

Corollary 4.6. *Let Ω be an open subset of \mathbb{R}^n . Let ω be a non-quasianalytic weight function satisfying condition B-M-M, and let $P = (P_j(D))_{j=1}^N$ be a system of order m satisfying condition (\mathcal{C}) . If*

$$(4.9) \quad \mathcal{E}_{(\omega)}^P(\Omega) \subseteq \mathcal{E}_{(\omega)}(\Omega),$$

or

$$(4.10) \quad \mathcal{E}_{\{\omega\}}^P(\Omega) \subseteq \mathcal{E}_{\{\omega\}}(\Omega),$$

then the system P is elliptic.

Proof. Beurling case:

Let us consider the system $Q = (D_j)_{j=1}^n$. Then $\mathcal{E}_{(\omega)}^Q(\Omega) = \mathcal{E}_{(\omega)}(\Omega)$ and (4.9) implies (4.2) with $\sigma(t) = \omega(t) = \omega(t^{\frac{r}{mh}})$ for $r = 1$ and $h = 1/m$.

By Theorem 4.4 we have that Q is $\frac{1}{m}$ -weaker than P , i.e.

$$\sum_{j=1}^n |\xi_j| \leq C \left(1 + \sum_{j=1}^N |P_j(\xi)| \right)^{\frac{1}{m}}, \quad \forall \xi \in \mathbb{R}^n.$$

This proves that the system P is elliptic, and hence the corollary is proved.

Roumieu case:

The proof is similar as in the Beurling case, using (4.10) and (4.3) instead of (4.9) and (4.2). \square

References

- [AJO] A.A. Albanese, D. Jornet, A. Oliaro, *Quasianalytic wave front sets for solutions of linear partial differential operators*, Integr. Equ. Oper. Theory 66 (2010), 153-181.
- [BG] C. Boiti, E. Gallucci, *The overdetermined Cauchy problem for ω -ultradifferentiable functions*, arXiv: 1511.07307, submitted for publication.
- [BJ1] C. Boiti, D. Jornet, *The problem of iterates in some classes of ultradifferentiable functions*, "Operator Theory: Advances and Applications", Birkhauser, Basel, **245** (2015), 21-33.
- [BJ2] C. Boiti, D. Jornet, *A characterization of the wave front set defined by the iterates of an operator with constant coefficients*, arXiv:1412.4954; submitted for publication.
- [BJ3] C. Boiti, D. Jornet, *A simple proof of Kotake-Narasimhan Theorem in some classes of ultradifferentiable functions*, to appear in J. Pseudo-Differ. Oper. Appl.
- [BJJ] C. Boiti, D. Jornet, J. Juan-Huguet, *Wave front set with respect to the iterates of an operator with constant coefficients*, Abstr. Appl. Anal. 2014, Article ID 438716 (2014), pp. 1-17.

- [BCM] P. Bolley, J. Camus, C. Mattera, *Analyticité microlocale et itérés d'opérateurs hypoelliptiques*, Séminaire Goulaouic-Schwartz, 1978-79, Exp N.13, École Polytech., Palaiseau.
- [BMM] J. Bonet, R. Meise, S.N. Melikhov, *A comparison of two different ways of define classes of ultradifferentiable functions*, Bull. Belg. Math. Soc. Simon Stevin 14 (2007), 425-444.
- [BC1] C. Bouzar, R. Chaili, *Une généralisation du problème des itérés*, Arch. Math. (Basel) 76 (1) (2001), 57-66.
- [BC2] C. Bouzar, R. Chaili, *A Gevrey microlocal analysis of multi-anisotropic differential operators*, Rend. Sem. Mat. Univ. Politec. Torino 64, n. 3 (2006), 305-317.
- [BMT] R.W. Braun, R. Meise, B.A. Taylor, *Ultradifferentiable functions and Fourier analysis*, Result. Math. 17 (1990), 206-237.
- [H1] L. Hörmander, *On interior regularity of the solutions of partial differential equations*, Comm. Pure Appl. Math. Vol. XI (1958), 197-218.
- [JH] J. Juan-Huguet, *Iterates and Hypoellipticity of Partial Differential Operators on Non-Quasianalytic Classes*, Integr. Equ. Oper. Theory 68 (2010), 263-286.
- [K1] H. Komatsu, *A characterization of real analytic functions*, Proc. Japan Acad. 36 (1960), 90-93.
- [K2] H. Komatsu, *Ultradistributions I. Structure theorems and a characterization*, J. Fac. Sci. Tokyo, Sec. IA 20 (1973), 25-105.
- [KN] T. Kotake, M.S. Narasimhan, *Regularity theorems for fractional powers of a linear elliptic operator*, Bull. Soc. Math. France 90 (1962), 449-471.
- [LM] J.L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, 3, Dunod, Paris (1970).
- [LW] G. Lysik, P. Wójcicki, *A characterization of elliptic operators*, Ann. Polon. Math. 111, n. 2 (2014), 145-148.
- [MV] R. Meise, D. Vogt, *Introduction to Functional Analysis*, Clarendon Press, Oxford (1997)
- [M] G. Métivier, *Propriété des itérés et ellipticité*, Comm. Part. Diff. Eq. 3 (9) (1978), 827-876.
- [NZ] E. Newberger, Z. Zielezny, *The growth of hypoelliptic polynomials and Gevrey classes*, Proc. Amer. Math. Soc. 39, n. 3 (1973), 547-552.
- [O] J. Oldrich, *Sulla regolarità delle soluzioni delle equazioni lineari ellittiche nelle classi di Beurling*, (Italian) Boll. Un. Mat. Ital. (4) 2 (1969), 183-195.
- [Z] L. Zanghirati, *Iterati di operatori e regolarità Gevrey microlocale anisotropa*, Rend. Sem. Mat. Univ. Padova 67 (1982), 85-104.

DIPARTIMENTO DI MATEMATICA E INFORMATICA, UNIVERSITÀ DI FERRARA, VIA MACHIAVELLI N. 30, I-44121 FERRARA, ITALY

E-mail address: chiara.boiti@unife.it

UNIVERSITY OF SCIENCES AND TECHNOLOGY OF ORAN, ALGERIA

E-mail address: rachidchaili@gmail.com; rachid.chaili@univ-usto.dz

UNIVERSITY IBN KHALDOUN OF TIARET, ALGERIA

E-mail address: mahrouz78@gmail.com